A New Element-Saving Equivalent Circuit for the Analysis of General Coupled n-Wire Transmission Lines

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Abstract —**A new efficient method for modeling the general coupled n-wire transmission line is proposed. The equivalent circuit presented in this paper has the following advantages: 1) no balanced lines are needed; 2) all element values are positive; and 3) the number of elements required by the equivalent circuit is at most equal to the number of nonzero couplings plus one. A simple method for deriving the equivalent circuit is also outlined. A few examples are supplied to show the application of the method.**

I. Introduction

IN work with planar meander line phase shifters, the need arises to analyze n parallel coupled transmission lines. This can be done by using the computer program COMPACT [1]. Since coupled n-wire transmission lines are not included in COMPACT component library, an equivalent circuit is needed for the coupled n-wire transmission line.

In recent years a few equivalent circuits for coupled n-wire transmission lines have been reported. Seviora [2] presented an equivalent circuit which required only single transmission lines, balanced as well as unbalanced. Grayzel [3] reported an equivalent circuit that in general consisted of nonsymmetrical coupled transmission line pairs. However, his method was restricted to structures with coupling between neighboring transmission lines only. Sato and Cristal [4] used transmission lines and short-circuited stubs for the synthesis of the port admittance matrix of the coupled transmission lines. Some applications of the methods in [2] and [4] are hampered by the fact that they include negative characteristic immittances.

The equivalent circuit presented in this paper will be a generalization of that by Grayzel [3]. A method for the derivation of an equivalent circuit of a general coupled n-wire transmission line is presented. It consists of symmetrical coupled transmission pairs (SCTL), the only type of coupled transmission line pair permitted in COMPACT [1], and, as will be shown later, a single transmission line.

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In Section II the theory is presented. This section consists of three parts. The first part is devoted to a parallel connection of coupled n-wire transmission line systems, and is included to form the necessary theoretical background. In the second part the derivation of the equivalent circuit is presented. Section II ends with a comparison between the equivalent circuit obtained by the method in this paper and the one presented by Sato and Cristal [4]. In Section III equivalent circuits are derived for a nonsymmetrical coupled transmission line pair as well as a coupled three-wire transmission line. The coupled transmission lines in this paper are restricted to be of commensurate electrical length.

II. Theory

A. Parallel Connection of Coupled n-Wire Transmission Line Systems

A coupled n-wire transmission line can be viewed as a $2n$-port (the ground plane is the $(n+1)$th wire. See Fig. 1.

In Fig. 1 $v_1, v_2, \cdots, v_n, v_{n+1}, v_{n+2}, \cdots, v_{2n}$ represent the port voltages with respect to the ground plane, and $i_1, i_2, \cdots, i_n, i_{n+1}, i_{n+2}, \cdots, i_{2n}$ the port currents. The transmission line system is described by the following admittance equation:

$$I = YV \tag{1}$$

where

$$I = [i_1 i_2 \cdots i_n ; i_{n+1} i_{n+2} \cdots i_{2n}]^t \tag{2}$$

$$V = [v_1 v_2 \cdots v_n ; v_{n+1} v_{n+2} \cdots v_{2n}]^t. \tag{3}$$

The admittance matrix, Y, can be expressed as

$$Y = \frac{1}{p}\begin{bmatrix} y & -\sqrt{(1-p^2)y} \\ -\sqrt{(1-p^2)y} & y \end{bmatrix} \tag{4}$$

where $p = \tanh(s\tau)$ is the Richards variable for the complex frequency s; τ is the commensurate one-way delay of the lines; and y is the $n \times n$ characteristic admittance matrix of the transmission line system.

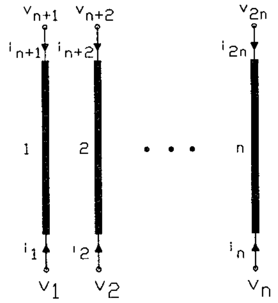

Fig. 1. n coupled transmission lines. (The ground plane is omitted from the figure.)

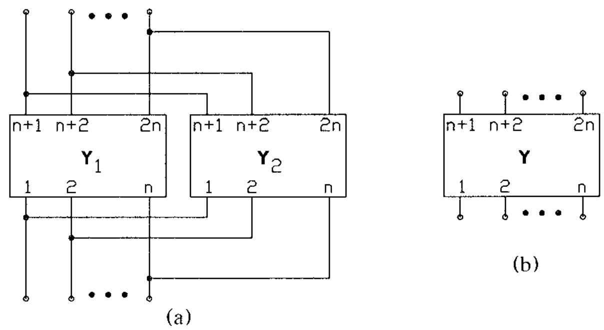

Fig. 2. (a) Two $2n$-ports connected in parallel. (b) Resulting $2n$-port.

Consider now a parallel connection of two $2n$-ports (Fig. 2(a).) It is known that the $2n$-ports in parts (a) and (b) of Fig. 2 are equivalent only if

$$\boldsymbol{Y} = \boldsymbol{Y}_1 + \boldsymbol{Y}_2. \tag{5}$$

Now let the $2n$-ports in Fig. 2(a) be coupled transmission line systems with characteristic admittance matrices $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, respectively. It is assumed that both transmission line systems have the same one-way delay, τ. The resulting $2n$-port will be a new coupled n-wire transmission line with a characteristic admittance matrix $\boldsymbol{y}$. For the two $2n$-ports in Fig. 2(a), $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ can be expressed through (4). Equations (4) and (5) are combined to give

$$\begin{bmatrix} y & -\sqrt{(1-p^2)y} & -\sqrt{(1-p^2)y} & y \end{bmatrix} = \begin{bmatrix} (y_1+y_2) & -\sqrt{(1-p^2)(y_1+y_2)} \\ -\sqrt{(1-p^2)(y_1+y_2)} & (y_1+y_2) \end{bmatrix} \tag{6}$$

From (6) it is seen that

$$\boldsymbol{y} = \boldsymbol{y}_1 + \boldsymbol{y}_2 \tag{7}$$

which is the rule of adding characteristic admittance matrices for parallel connected coupled transmission line systems.

B. Derivation of the Equivalent Circuit

It is assumed that the coupled n-wire line has a characteristic admittance matrix

$$\boldsymbol{y} = \begin{bmatrix} y_{11} & -y_{12} & \cdots & -y_{1n} \\ -y_{21} & y_{22} & \cdots & -y_{2n} \\ \vdots & \vdots & & \vdots \\ -y_{n1} & y_{n2} & \cdots & -y_{nn} \end{bmatrix}. \tag{8}$$

This matrix is symmetric and hyperdominant, that is,

$$y_{ij} \geqslant 0 \qquad \text{for } i=1,\cdots,n; \quad j=1,\cdots,n \tag{9a}$$

$$y_{ii} \geqslant \sum_{\substack{j=1 \\ j\neq i}}^{n} y_{ij} \qquad \text{for } i=1,\cdots,n \tag{9b}$$

$$y_{ij} = y_{ji}. \tag{9c}$$

It is also assumed that all lines couple to at least one of the other lines.

The equivalent circuit is now derived by repeating the following steps $n-1$ times, until a single transmission line remains.

Step 1: The characteristic admittance from conductor i to ground y_{0i} is calculated for all lines:

$$y_{0i} = y_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} y_{ij}. \tag{10}$$

Step 2: The lines are now renumbered in descending order of y_{0i}:

$$y_{01} \geqslant y_{02} \geqslant \cdots \geqslant y_{0,n-1} \geqslant y_{0n}. \tag{11}$$

Step 3: The coupling from line n to line i, $i=1,\cdots,n-1$ is represented by an SCTL with the characteristic admittance matrix y^{ni}:

$$\boldsymbol{y} = \begin{bmatrix} y_s^{ni} & -y_m^{ni} \\ -y_m^{ni} & y_s^{ni} \end{bmatrix}. \tag{12}$$

The contribution from this SCTL to the characteristic admittance matrix of the n-wire line is $\bar{y}^{ni}$, where

$$\bar{\boldsymbol{y}}^{ni} = \begin{bmatrix} \bar{y}_{11}^{ni} & -\bar{y}_{12}^{ni} & \cdots & -\bar{y}_{1n}^{ni} \\ -\bar{y}_{21}^{ni} & \bar{y}_{22}^{ni} & \cdots & -\bar{y}_{2n}^{ni} \\ \vdots & \vdots & & \vdots \\ -\bar{y}_{n1}^{ni} & \bar{y}_{n2}^{ni} & \cdots & -\bar{y}_{nn}^{ni} \end{bmatrix}. \tag{13}$$

The elements $\bar{y}_{kl}^{ni}$, $k=1,\cdots,n$, $l=1,\cdots,n$, can be expressed as

$$\bar{y}_{kl}^{ni} = y_s^{ni}\delta_{kl}(\delta_{ki}+\delta_{kn}) + y_m^{ni}(\delta_{ki}\delta_{ln}+\delta_{kn}\delta_{li}) \tag{14}$$

where δ_{ij} is the Kronecker delta, defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j. \end{cases} \tag{15}$$

All couplings from line n are now considered. The other couplings between the lines $1, 2, \cdots, n-1$ are included in the reduced characteristic admittance matrix $\boldsymbol{y}^r$. The characteristic admittance matrix, $\boldsymbol{y}$, of the n-wire line can now be expressed as

$$y = y^r + \sum_{i=1}^{n-1} \bar{y}^{ni} \tag{16}$$

The reduced matrix, y^r, is choosen so that

$$y_{ni}^r = y_{in}^r = 0, \qquad i = 1,\cdots,n \tag{17}$$

$$y_{ij}^r = y_{ij}, \qquad i \neq j,\ i = 1,\cdots,n-1,\ j = 1,\cdots,n-1. \tag{18}$$

The mutual characteristic admittances are retrieved from (16) using (14) and (17):

$$y_m^{ni} = y_{ni}, \qquad i = 1,\cdots,n-1 \tag{19}$$

while y_s^{ni} is calculated in the next step.

Step 4: By using (14) and (17), the following equation for y_s^{ni} can be derived from (14):

$$y_{nn} = \sum_{i=1}^{n-1} y_s^{ni}. \tag{20}$$

Equation (20) is solved for y_s^{ni} by the method below. The assumption made above implies that at least one of the y_{nj} $(i \neq n)$ is nonzero. One of these is chosen and called y_{nI}. The quantity t_i is defined as

$$t_i = \frac{y_{ni}}{y_{nI}}, \qquad i = 1,\cdots,n-1. \tag{21}$$

Now the self characteristic admittances y_s^{ni} are calculated from

$$y_s^{ni} = t_i \frac{y_{nn}}{\sum_{j=1}^{n-1} t_j}, \qquad i = 1,\cdots,n-1. \tag{22}$$

By the use of (21), (9b) can be rewritten as

$$y_{nn} \geqslant y_{nI} \sum_{j=1}^{n-1} t_j. \tag{23}$$

Combining (22) and (23) gives

$$y_s^{ni} \geqslant t_i y_{nI}, \qquad i = 1,\cdots,n-1. \tag{24}$$

By the use of (19) and (21), (24) can be rewritten as

$$y_s^{ni} \geqslant y_m^{ni}, \qquad i = 1,\cdots,n-1. \tag{25}$$

The inequality (25) implies that all $n-1$ SCTL's are physically realizable.

Step 5: The diagonal elements in rows 1 to $n-1$ in y^r can be expressed as

$$y_{ii}^r = y_{ii} - y_s^{ni}, \qquad i = 1,\cdots,n-1. \tag{26}$$

A new characteristic admittance matrix, y', of order $n-1$ is formed by deleting the nth row and column in y^r. From (11),

$$y_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} y_{ij} \geqslant y_{nn} - \sum_{j=1}^{n-1} y_{nj}, \qquad i = 1,\cdots,n-1. \tag{27}$$

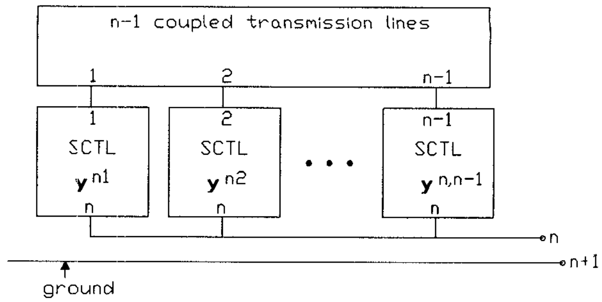

Fig. 3. Equivalent circuit after first cycle.

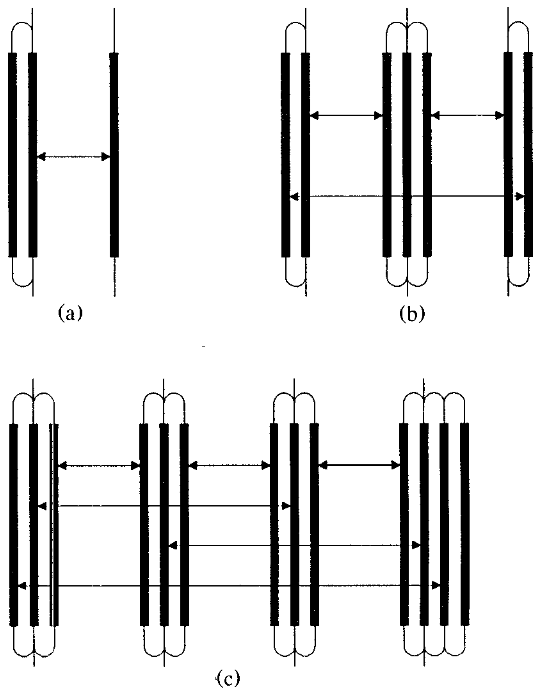

Fig. 4. The equivalent circuit for (a) two, (b) three, and (c) four coupled transmission lines.

Equation (27) can also be rewritten in the following form:

$$y_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} y_{ij} \geqslant \sum_{j=1}^{n-1} \left(y_s^{nj} - y_m^{nj} \right), \qquad i = 1,\cdots,n-1. \tag{28}$$

All terms in the right-hand expression are nonnegative:

$$y_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} y_{ij} \geqslant y_s^{ni} - y_m^{ni}, \qquad i = 1,\cdots,n-1. \tag{29}$$

Rearranging (29) and then using (19) and (26) gives

$$y_{ii}' \geqslant \sum_{\substack{j=1 \\ j \neq i}}^{n-1} y_{ij}, \qquad i = 1,\cdots,n-1. \tag{30}$$

Inequality (30) implies that the reduced characteristic admittance matrix, y', has to fulfill the same requirements assumed for the original admittance matrix.

After performing the first cycle of steps 1 to 5, an equivalent circuit of the form shown in Fig. 3 is obtained. For simplicity, only one end of the n-wire line is shown in this figure.

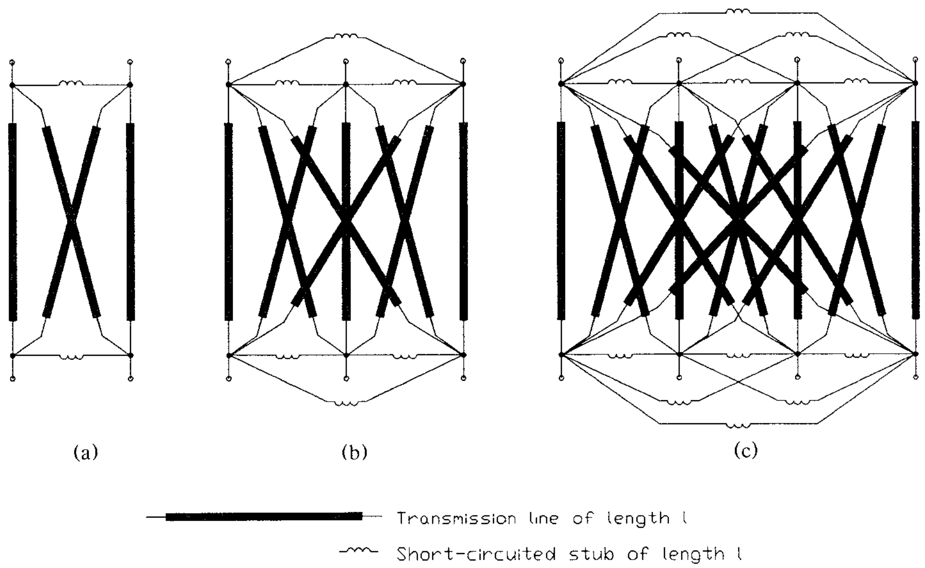

Fig. 5. The equivalent circuit of Sato and Cristal [4] for (a) two, (b) three, and (c) four coupled transmission lines. (d) Symbols.

TABLE I
GENERAL CASE OF COUPLED LINES

	Sato and Cristal [4]			This Paper		
Number of Coupled Lines	Number of Transmission Lines	Number of Short-Circuited Stubs	Total	Number of SCTL's	Number of Transmission Lines	Total
2	4	2	6	1	1	2
3	9	6	15	3	1	4
4	16	12	28	6	1	7
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
n	n^2	n^2-n	$2n^2-n$	$(n^2-n)/2$	1	$(n^2-n+2)/2$

Steps 1 to 5 are repeated until step 5 gives a characteristic admittance matrix of order 1 (a single transmission line).

C. Comparison with the Equivalent Circuit Presented by Sato and Cristal

In the general coupled n-wire transmission line (the ground plane being the $(n+1)$th wire), there are $(n^2-n)/2$ couplings. Each coupling is realized in the equivalent circuit as an SCTL. A single transmission line is also required in general. This means that the number of elements needed in the equivalent circuit is at most $(n^2-n+2)/2$.

Examples of equivalent circuits for two, three, and four coupled transmission lines are shown in Fig. 4. The equivalent circuits by Sato and Cristal [4] for two, three, and four coupled transmission lines are shown in Fig. 5.

In Table I is shown a comparison of the number of elements for the general case, and in Table II is shown the case when only neighboring lines are coupled. From these tables it is observed that the equivalent circuit derived in subsection II-B always requires less than a third of the components required by the equivalent circuit of Sato and Cristal [4].

III. NUMERICAL EXAMPLES

In this section two numerical examples are presented. The aim of the examples is to show how the method presented in subsection II-B is applied. First an equivalent circuit for a nonsymmetrical directional coupler is derived. As the second example a meander line consisting of three coupled transmission lines is shown.

A. Nonsymmetrical Direction Coupler

Cristal has designed an experimental matched −10 dB directional coupler with a center frequency 1.5 GHz and an impedance transformation from 50 Ω to 75 Ω [5]. Its circuit is shown in Fig. 6. From the numerical data presented in [5], the characteristic admittance matrix can be calculated to be

$$y = \begin{bmatrix} 21.09 & -5.45 \\ -5.45 & 14.06 \end{bmatrix}$$

where the matrix values are given in mS. The design of

TABLE II
COUPLING ONLY BETWEEN NEIGHBORING LINES

	Sato and Cristal [4]			This Paper		
Number of Coupled Lines	Number of Transmission Lines	Number of Short-Circuited Stubs	Total	Number of SCTL's	Number of Transmission Lines	Total
2	4	2	6	1	1	2
3	7	6	15	3	1	4
4	10	12	28	6	1	7
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
n	$3n-2$	$2(n-1)$	$5n-4$	$n-1$	1	n

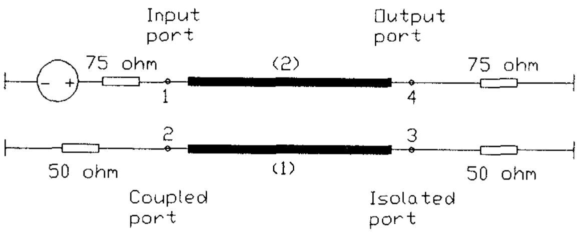

Fig. 6. Nonsymmetrical directional coupler.

the equivalent circuit is performed through the following steps.

Step 1: From (10),

$$y_{01} = 15.64 \text{ mS}$$

$$y_{02} = 8.61 \text{ mS}.$$

Step 2: The lines are arranged so that

$$y_{01} > y_{02}.$$

Step 3: The mutual characteristic admittance is found from (19) to be

$$y_m^{21} = 5.45 \text{ mS}.$$

Step 4: The self characteristic admittance is calculated by the use of (22). Note from (21) that $t_1 = 1$:

$$y_s^{21} = 14.06 \text{ mS}.$$

In this case the result is obvious because only one SCTL is needed to realize a coupled two-wire line.

Step 5: From (26), y_{11} in the original characteristic admittance matrix is replaced by

$$y'_{11} = 7.03 \text{ mS}.$$

The original characteristic admittance matrix is replaced by a reduced characteristic admittance matrix of order 1 with

$$y' = y'_{11}.$$

This characteristic admittance matrix should be interpreted as the characteristic admittance, Y, for a single transmission line

$$Y = y' = 7.03 \text{ mS}.$$

In COMPACT, a single transmission line is described by its characteristic impedance:

$$Z = \frac{1}{Y} = 142.2\ \Omega.$$

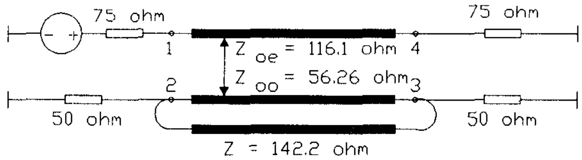

Fig. 7. The equivalent circuit for the nonsymmetrical directional coupler shown in Fig. 6.

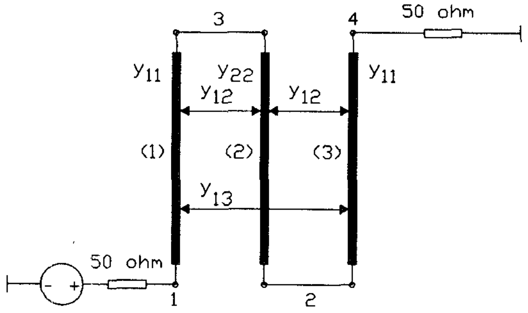

Fig. 8. Meander line with three coupled lines.

SCTL's are described by their even- and odd-mode characteristic impedances:

$$Z_{0e} = \frac{1}{Y_{0e}} = \frac{1}{y_s^{21} - y_m^{21}} = 116.1\ \Omega$$

$$Z_{0o} = \frac{1}{Y_{0o}} = \frac{1}{y_s^{21} + y_m^{21}} = 51.26\ \Omega.$$

The equivalent circuit is shown in Fig. 7, where the single transmission line is needed to account for the difference between the self characteristic admittances of the two lines.

B. Three Coupled Transmission Lines

Rehnmark [6], [7] analyzes the circuit shown in Fig. 8. As a second example the following characteristic admittance matrix is chosen ([6, table 4:1, row 30] or [7, table 4:1, row 30]):

$$\mathbf{y} = \begin{bmatrix} 22.54 & -8.84 & -0.2 \\ -8.84 & 27.36 & -8.84 \\ -0.2 & -8.84 & 22.54 \end{bmatrix} \text{mS}.$$

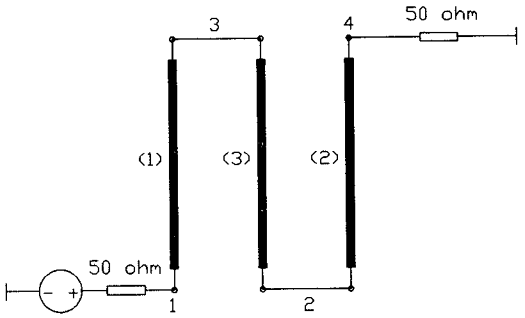

Fig. 9. The circuit in Fig. 8 with renumbered lines.

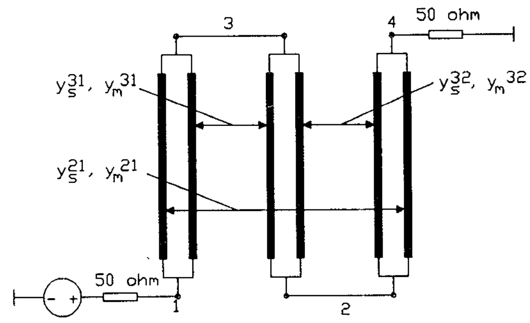

Fig. 10. Equivalent circuit for the meander line shown in Fig. 8.

An equivalent circuit is now designed by the above procedure.

Step 1: From (10),

$$y_{01} = 13.50 \text{ mS}$$

$$y_{02} = 9.68 \text{ mS}$$

$$y_{01} = 13.50 \text{ mS}.$$

Step 2: It is observed from the values of y_{01}, y_{02}, and y_{03} that

$$y_{01} = y_{03}$$

$$y_{03} \geqslant y_{02}.$$

It follows that the required renumbering is

$$2 \rightarrow 3 \rightarrow 2.$$

In Fig. 9 the lines are renumbered. The transmission line system in this figure has the characteristic admittance matrix

$$y = \begin{bmatrix} 22.54 & -0.2 & -8.84 \\ -0.2 & 22.54 & -8.84 \\ -8.84 & -8.84 & 27.36 \end{bmatrix} \text{mS}.$$

Step 3: The mutual characteristic admittances y_m^{31} and y_m^{32} are determined with the use of (19) as

$$y_m^{31} = 8.84 \text{ mS}$$

$$y_m^{32} = 8.84 \text{ mS}$$

Step 4: y_{31} is chosen as y_{nl}. Then t_1 and t_2 become (see (21))

$$t_1 = 1$$

$$t_2 = 1.$$

The self-admittances can now be calculated from (22):

$$y_s^{31} = 13.68 \text{ mS}$$

$$y_s^{32} = 13.68 \text{ mS}.$$

Step 5: The diagonal elements y_{11} and y_{22} are changed to (see (26))

$$y'_{11} = 8.86 \text{ mS}$$

$$y'_{22} = 8.86 \text{ mS}.$$

The reduced $\boldsymbol{y}$ matrix, $\boldsymbol{y}'$, then becomes

$$y' = \begin{bmatrix} 8.86 & -0.2 \\ -0.2 & 8.86 \end{bmatrix} \text{mS}.$$

The characteristic admittance matrix $\boldsymbol{y}'$ represents an SCTL (compare (12)). The last SCTL is found to be

$$y_m^{21} = 0.2 \text{ mS}$$

$$y_s^{21} = 8.86 \text{ mS}.$$

The equivalent circuit is shown in Fig. 10. The example shown in this figure is independent of the way the six line terminations are interconnected (in this case a meander line was chosen). The outlined method can of course be applied to any interconnected n-wire transmission line.

IV. Conclusion

The equivalent circuit presented in this paper for a coupled n-wire transmission line requires at most $(n^2 - n + 2)/2$ elements. As observed by the numerical examples presented, the equivalent circuit is easy to derive. It does not require the negative characteristic immittances required by, for instance, the method of Sato and Cristal [4].

Acknowledgment

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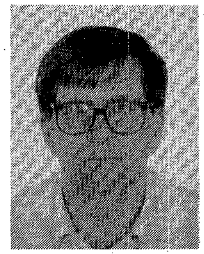

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